Int J Theor Phys (2008) 47: 859–864
DOI 10.1007/s10773-007-9510-3

# Fundamental Equations of Quantum Mechanics in Time-Varying Domain

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Received: 18 April 2007 / Accepted: 25 July 2007 / Published online: 11 September 2007


**Abstract** Fundamental equations of quantum mechanics in time-varying domain are presented. The used method consists in transforming the variable domain into a fixed domain. The transformation has to be covariant in relation to the wavefunction. The new fundamental equations turn out to be a generalization of the classical equations established in a Newtonian space-time. When the time-varying domain becomes stationary, we find again the fundamental equations of the classical quantum mechanics.

**Keywords** Wave equation · Wave function · Quantum mechanics · Time-varying boundary value problems · Time-varying domains

## 1 Introduction

The theory of wave functions and fundamental equations of quantum mechanics is a well-known issue in text-books [15]. It is also well-known, that this theory has been established in Newtonian space-time that is when the space and time coordinates are independent. What happens? This is the main question to be addressed when considering a time-varying space that is when the space and time coordinates are interdependent. This kind of problems is of interest for instance in time-varying boundary systems [2, 10–12, 14], in expanding forces fields [4] and in the evolution of meta-stable states in the early universe that is an interesting issue in cosmology [9]. In general relativity, "space" itself changes in time, effectively the distance between two points can change from one instant to the next. Beyond all consideration, the issue is to provide qualitative answer for description of particle evolution. In general, the solution of such problems amounts to employing numerical techniques or more general mathematical methods [3, 5, 7, 8].

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The aim of this work is, therefore, to investigate the possible formulation of the fundamental equations of particle evolution in time-varying domain. The method consists in transforming the associated wave function for the variable domain to an associated wave function of the covariant form in a fixed domain. We find that the transformation is constituted by coupled linear partial equations. We have also determined the new formulation of the fundamental equations. This formulation turns out to be a generalization of the classical formulation.

The paper is organized as follows: Sect. 2 proposes a covariant transformation of the wave function. Then, in Sect. 3, we give the principal equations. Finally, the results are discussed in Sect. 4.

## 2 Determination of the Wave Function

### 2.1 Formulation of the Problem

Let be a particle of momentum $P$ and energy $E$ propagating with a velocity $V$ in the O$x$ direction. Assume that the domain of propagation changes in time, in the same direction, with the moving law $x = a(\tau)$.

According to the de Broglie hypothesis, at a fundamental level, both radiation and matter have characteristics of particles and waves. In comparison with the works published previously [5–7], we assume that the associated wave function to the particle, propagating in the same direction, expresses in terms of functional Fourier's series. For convenience and simplicity, we consider only the one-dimensional case. Thus, for one term, the associated wave function, in the time-varying domain, can then be written as follows:

$$\psi(x,t) = A \exp ik\varphi(x-\tau) \tag{2.1}$$

where $k$ is the wave number and $\tau = V.t$.

The initial and boundary conditions vary with the respective applications and are specified there.

Equation (2.1) shows that the associated wave function $\psi(x,\tau)$ can only be determined if the function $\varphi$ is known.

### 2.2 Proprieties of $\varphi$

$\varphi$ is a complex transformation function which must leave the wave function form invariant. We are interested in those covariant transformations which map every point of a given time-varying domain $R(t,x)$ into a point of a fixed domain $S(\xi,\eta)$. Therefore, $\varphi(x,t)$ is, in general, of the form

$$\varphi(x,t) = \xi(x,t) + i\eta(x,t). \tag{2.2}$$

The complex transformation (2.2) is mathematically equivalent to the two real transformations functions

$$\xi = \xi(x,t), \qquad \eta = \eta(x,t), \tag{2.3}$$

with the corresponding real, inverse transformations

$$x = f(\eta,\xi), \qquad t = g(\eta,\xi). \tag{2.4}$$

The transformation (2.3) can only be determined if

$$\frac{D(\eta,\xi)}{D(x,t)} \neq 0. \tag{2.5}$$

Thanks to the complex transformation (2.2) or the equivalent real transformations (2.3), the associated wave function $\psi(x,\tau)$ of the variable domain becomes an associated wave function $\psi(\eta,\xi)$ of the fixed domain. Hence,

$$\psi(x,\tau) \equiv \psi(\eta,\xi). \tag{2.6}$$

Therefore, the associated wave function $\psi(\eta,\xi)$ of the fixed domain writes:

$$\psi(\eta,\xi) = A \exp ik(\eta - \xi) \tag{2.7}$$

where $k$ is the wave number in the fixed domain $S(\xi,\eta)$.

It is clear that the possible transformation functions $\varphi(x,t)$ depends on both the conditions of covariance and the moving law $x = a(t)$.

Finally, we have to point out that $\psi(\eta,\xi)$ and $\psi(x,\tau)$ are solutions of wave equations in stationary and non-stationary domain respectively.

## 2.3 Corollary

From (2.1), (2.5) and (2.7), we deduce the following expressions:

$$\varphi(x-\tau) = \eta - \xi, \tag{2.8}$$

$$\varphi(x+\tau) = \eta + \xi. \tag{2.9}$$

Equations (2.8) and (2.9) yield the relationships between the new coordinates $(\xi,\eta)$ and the old coordinates $(x,\tau)$ such that:

$$\eta = \frac{1}{2}[\varphi(x+\tau) + \varphi(x-\tau)], \tag{2.10}$$

$$\xi = \frac{1}{2}[\varphi(x+\tau) - \varphi(x-\tau)]. \tag{2.11}$$

We note that at $\tau = 0$, the two domains are stationary. Thus, the transformation $\varphi$ becomes an identical transformation.

Under the assumption of existence of the first and second derivatives, the relations (2.10) and (2.11) yield the following conditions:

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial\tau} \quad \text{and} \quad \frac{\partial\xi}{\partial\tau} = \frac{\partial\eta}{\partial x}. \tag{2.12}$$

The above fundamental relations show that the complex function (2.2) is not analytic.

In addition to (2.12), the transformation functions are solutions of wave equations

$$\frac{\partial^2\xi}{\partial x^2} = \frac{\partial^2\xi}{\partial\tau^2} \quad \text{and} \quad \frac{\partial^2\eta}{\partial x^2} = \frac{\partial^2\eta}{\partial\tau^2}. \tag{2.13}$$

Thus, the transformation functions which accomplish the mapping of the variable domain into a fixed domain can be determined by solving a functional equation whether the boundary motion $x = a(t)$ is known [1]. However, other methods exist and can also be used for instance the conformal mapping [5, 13].

It is interesting to note that all the above results are in good agreement with those find previously [5]. As illustration, we find in this last reference explicit results regarding the function transformation between a domain with a linear boundary motion and a fixed band.

## 3 Equation of Evolution

### 3.1 Non-Relativistic Equation

According to quantum mechanics, the wave description of de Broglie is not adequate to explain the properties of particles. Then, we follow the Schrödinger approach by using the associated wave function (2.1) and the nonrelativistic classical energy of the particle given by

$$E = U + K = U + \frac{p^2}{2m} \tag{3.1}$$

where $E$ is the total energy. $U$ is the potential energy, and $K$ is the kinetic energy. $P$ and $m$ are the momentum and mass respectively.

The derivatives of the associated wavefunction (2.1) with respect to time and position are:

$$\frac{\partial \psi}{\partial t} = -ikV\varphi'\psi = -i\omega\varphi'\psi, \tag{3.2}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2\left[\varphi'^2 - \frac{i}{k}\varphi''\right]\psi, \tag{3.3}$$

$$\frac{\partial \psi}{\partial x} = ik\varphi'\psi, \tag{3.4}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2\left[\varphi'^2 - \frac{i}{k}\varphi''\right]\psi. \tag{3.5}$$

According to the relation $\omega = E/\hbar$, (3.2) is then

$$\frac{\partial \psi}{\partial t} = -i\varphi'\frac{E}{\hbar}\psi. \tag{3.6}$$

Also according to de Broglie relation, $k = P/\hbar$ so that

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{P^2}{\hbar^2}\left[\varphi'^2 - \frac{i}{k}\varphi''\right]\psi. \tag{3.7}$$

Note that $\varphi'$ is the first derivative of $\varphi(x - V.t)$.

To relate (3.1) with the derivatives (3.6) and (3.7), multiply the conservation energy equation by $\Psi$. Thus, we find the following equation:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2 k\varphi'}{2m[k\varphi'^2 - i\varphi'']}\frac{\partial^2 \psi}{\partial x^2} + \varphi' U\psi. \tag{3.8}$$

Equation (3.8) is a fundamental equation for describing quantum behaviour of physical system evolving over time in time-varying domain. Actually, it is the generalized Schrödinger's equation. Indeed, if the domain becomes stationary, the transformation functions $\varphi(x,t)$ becomes identity. Therefore, we obtain $\varphi' = 1$ and $\varphi'' = 0$ so that we find again the famous Schrödinger's equation, namely,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U\psi. \tag{3.9}$$

### 3.2 Relativistic Equations

In this case, the relativistic energy is given by the well-known relation

$$E^2 = m^2c^4 + c^2P^2. \tag{3.10}$$

From (3.3) and (3.7) and according to the relation $\omega = E/\hbar$, we can write

$$-\frac{\hbar^2 k}{k\varphi'^2 - i\varphi''}\frac{\partial^2\psi}{\partial t^2} = E^2\psi, \tag{3.11}$$

$$-\frac{kc^2\hbar^2}{k\varphi'^2 - i\varphi''}\frac{\partial^2\psi}{\partial x^2} = c^2P^2\psi. \tag{3.12}$$

By combining (3.10), (3.11) and (3.12), we get the following equation:

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2 k}[k\varphi'^2 - i\varphi'']\psi = 0. \tag{3.13}$$

It is obvious that (3.13) represents the generalized Klein-Gordon equation. Indeed, for a stationary domain ($\varphi' = 1$ and $\varphi'' = 0$), we find again the classical Klein-Gordon equation, namely,

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{3.14}$$

In the same way, we determine the generalized Dirac's equation, namely,

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\alpha_1\frac{\partial\psi}{\partial x} + \varphi'\beta mc^2\psi. \tag{3.15}$$

It should be stressed that $\varphi' = 1$ when the space becomes stationary so that we find again the classical Dirac's equation.

## 4 Conclusion

In order to formulate a theory of quantum mechanics in variable space paralleling the classical theory, we have proposed a functional wave function. This functional wave function may be assumed to have a form similar to that of the wave function of classical quantum mechanics but must be so designed as to harmonize with the Newtonian space-time in the limit case when space motion is negligible. It is interesting to note that the covariance of the wavefuction leads to fundamental conditions which must be satisfied by transformation functions.

Thus, suitable partial differential equations are determined according to the Schrödinger approach. These equations turn out to be the generalization of the classical equations, namely, time-dependent Schrödinger's equation, Klein-Gordon's equation and Dirac's equation.

The author thinks that other works are necessary for interpreting and developing this approach. Indeed, the mathematical significance of this approach is obvious. However, to validate the equations and interpret their outlooks, some applications are needed. It is clear that, the fact of expressing the equations in time-varying domain can imply that there will be any space-time dependencies or correlations. Furthermore, state vectors or operators can be affected and may be lead to any observable space-time effects. We can, also, extend this approach to more one-dimension and to relativity equations. Finally, may be all these considerations can lead to a possible link between quantum mechanics and general relativity.

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